The effect of superluminal phase velocity on electron acceleration in a powerful electromagnetic wave

A.P.L.Robinson[1], A.V.Arefiev[2],V.N.Khudik[2]

Abstract
We have derived an analytic solution for the problem of a single electron in a EM plane wave of arbitrary strength and arbitrary phase velocity. The solution has been checked against direct numerical integration. From this analytic solution we can begin to understand the extent to which EM dispersion (and thus superluminal phase velocities) due either plasma dispersion or wave-guiding affects DLA.

[1] *Plasma Physics Group, Central Laser Facility, STFC Rutherford-Appleton Laboratory, Didcot, OX11 0QX, UK*
[2] *Institute for Fusion Studies, The University of Texas, Austin,Texas 78712, USA*

Contents

1. Introduction

Here we report on our recent work on problems in Direct Laser Acceleration (DLA), soecifically on single electron motion in strong EM waves. We have addressed the problem of motion in EM waves which have an arbitrary phase velocity. This is highly relevant to laser-plasma interactions where both plasma dispersion and wave-guiding effects can lead to superluminal phase velocities. We have derived a solution to the problem of a single electron in a strong EM *plane* wave with arbitrary phase velocity, and we quantify the extent to which superluminal phase velocities can have a deleterious effect on DLA in the single electron, plane wave case. Our solution is checked against direct numerical integration of the equations of motion in primitive form.

2. Analytic Solution of Electron Motion in Laser Field with Arbitrary Phase Velocity

To understand the effect that a superluminal phase velocity has on electron acceleration, we can look at the motion of a single electron in a plane EM wave. To this end, we derive an analytical solution for the momentum of a single electron accelerated by a plane electromagnetic wave with a given phase velocity v_p. Let the plane wave be described by a vector potential of the form,

$$\mathbf{A} = [0, A, 0] = [0, A_0 \cos(\omega_L \tau), 0], \tag{1}$$

where $\tau = t - x/v_p$, and ω_L is the frequency of the field. The electric and magnetic fields are related to the vector potential via $\mathbf{E} = -\partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, so the electric field of this wave is polarized in the y-direction, and the magnetic field in the z-direction. Note that we have introduced the dispersive nature of electromagetic waves in plasmas via the specification of an arbitrary phase velocity, v_p, instead of c. Only this needs to be specified, and the actual dispersion relation itself does not.

The equations of motion of the electron that need to be considered are:

$$\frac{dp_x}{dt} = -|e| v_y B_z, \tag{2}$$
$$\frac{dp_y}{dt} = -|e| E_y + |e| v_x B_z, \tag{3}$$
$$\frac{dp_z}{dt} = 0, \tag{4}$$
$$\frac{d\gamma}{dt} = -\frac{|e| v_y E_y}{m_e c^2}. \tag{5}$$

From the definition of τ, one can differentiate to obtain,

$$\frac{d\tau}{dt} = 1 - \frac{v_x}{v_p}, \tag{6}$$

and this can then be used to write the field components as $E_y = -\partial_\tau A$, $B_z = (1/v_p)\partial_\tau A$. In turn this means that equations (2), (3), and (5) can be re-written in the form,

$$\frac{dp_x}{dt} = \frac{|e| v_y}{v_p} \frac{dA_y}{d\tau}, \tag{7}$$
$$\frac{dp_y}{dt} = |e| \frac{dA_y}{d\tau} \left(1 - \frac{v_x}{v_p}\right), \tag{8}$$
$$\frac{d\gamma}{dt} = \frac{|e| v_y}{m_e c^2} \frac{dA_y}{d\tau}. \tag{9}$$

Using Eqs. (6) and (8), one can immediately obtain

$$p_y = |e|A_y, \tag{10}$$

which is also obtained in the case where $v_p = c$. The next constant of motion is obtained from combining Eqs. (7) and (9). This yields,

$$\gamma - \frac{v_p}{c}\frac{p_x}{m_e c} = 1, \tag{11}$$

provided that the electron is at rest prior to the interaction with the laser pulse. We now need to relate this to the dephasing rate, i.e. $d\tau/dt$. On multiplying Eq. (6) with γ one obtains,

$$\gamma\frac{d\tau}{dt} = \gamma - \frac{p_x}{m_e v_p}. \tag{12}$$

In the case where $v_p = c$, the right-hand side becomes exactly equal to unity. However, in the general case of $v_p \neq c$ we instead have,

$$\gamma\frac{d\tau}{dt} = 1 + \left(\frac{v_p}{c} - \frac{c}{v_p}\right)\frac{p_x}{m_e c}. \tag{13}$$

Adopting the normalizations $\tilde{p}_x = p_x/m_e c$ and $a_y = |e|A_y/m_e c$, we can re-write Eq.(7) as,

$$\frac{d\tilde{p}_x}{dt} = \frac{c}{\gamma v_p}\frac{d}{d\tau}\left(\frac{a_y^2}{2}\right), \tag{14}$$

and we can use Eq. (13) to both make a change of variables (from t to τ on the left-hand side) and eliminate γ. The resulting equation can be directly integrated to obtain,

$$\tilde{p}_x + \left(\frac{v_p}{c} - \frac{c}{v_p}\right)\frac{\tilde{p}_x^2}{2} = \frac{c}{2v_p}a_y^2. \tag{15}$$

It follows directly from Eq. (15) that the longitudinal momentum of a forward moving electron is

$$\frac{p_\parallel}{m_e c} = \tilde{p}_x = \frac{\sqrt{u^2 + a^2(u^2-1)} - u}{u^2 - 1}, \tag{16}$$

where for compactness we have introduced

$$u \equiv v_p/c. \tag{17}$$

For $u \to 1$, we recover from Eq. (16) the well-known result for the luminal case,

$$\frac{p_\parallel}{m_e c} = \frac{a^2}{2}. \tag{18}$$

It is evident from Eq. (16) that the superluminosity is not important as long as $a^2(u^2-1) \ll u^2$. This condition is equivalent to the condition $\delta u \ll 2/a^2$ for $a \gg 1$, where again $\delta u = v_p/c - 1$.

For $\delta u \gg 2/a^2$, we find from Eq. (16) that

$$\frac{p_\parallel}{m_e c} \approx \frac{a}{\sqrt{2\delta u}}, \tag{19}$$

where we assumed that $\delta u \ll 1$. We therefore conclude that the superluminosity leads to a reduction of the maximum longitudinal electron momentum. More importantly, $p_\parallel$ scales linearly with wave amplitude for $\delta u \gg 2/a^2$.

The change in the scaling for the longitudinal momentum with the wave amplitude has a profound effect on the maximum longitudinal velocity the electron can achieve. For $\delta u \ll 1/2a^2$, the parallel velocity approaches the speed of light with the increase of the wave amplitude:

$$(c - v_\parallel)/c = 1 - p_\parallel/\gamma m_e c \approx 2/a^2. \tag{20}$$

However, once a exceeds the critical value of $1/\sqrt{2\delta u}$ the maximum longitudinal velocity becomes independent of the wave amplitude:

$$(c - v_\parallel)/c = 1 - p_\parallel/\gamma m_e c \approx \delta u. \tag{21}$$

This result indicates that the maximum longitudinal electron velocity is no longer limited by c, but rather by $c(1-\delta u)$ due to the superluminosity of the accelerating wave. This feature unavoidably manifests itself at high field amplitudes for a given superluminal phase velocity.

3. Numerical Calculations

In this section we present some example numerical calculations which both illustrate the general results found in the previous sections and provide a numerical test of those findings. In these calculations we integrate the primitive forms of the equations of motion [Eqs. (2) and (3)] with a specified laser field and the equations for the spatial coordinates of the electron. The algorithm used for this is the second order centered method of Boris. The time step was chosen to take the magnitude of the vector potential into account [**?**]. The normalized vector potential of the laser pulse in all simulations had the form

$$a_y = a_0\cos(\omega_L\tau)\exp\left[\frac{-(x - v_g t - x_0)^2}{2c^2 t_L^2}\right], \tag{22}$$

where t_L is a laser pulse duration in vacuum, and v_g is the group velocity of the pulse (= c in vacuum).

In order to test our analytical solution we carried out a series of calculations in which we varied a_0 and v_p and compared the results to the predictions of Eq. (16) by comparing the peak $p_x/m_e c$ obtained. For the series in which we varied only a_0, we used v_p =1.15c and v_g =0.866c for the phase and group velocities. For the laser pulse we used λ_L =1μm and t_L =20 fs throughout.

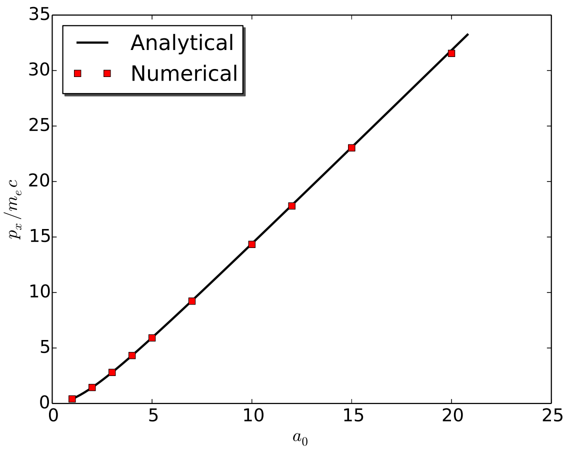

Figure 1. Peak longitudinal momentum p_x from a series of numerical calculations in which a_0 is varied for $v_p = 1.15c$ against the predictions of Eq. (16).

The results of the first series are plotted in Fig. 1 against the predictions of Eq. (16). As can be seen, the agreement is excellent. Figure 2 shows that if the phase velocity is varied for constant a_0 ($a_0 = 5$ was used) then we also obtain excellent agreement with the analytic solution. We can therefore conclude that the analytic solution has been reasonably verified by comparison to numerical calculations.

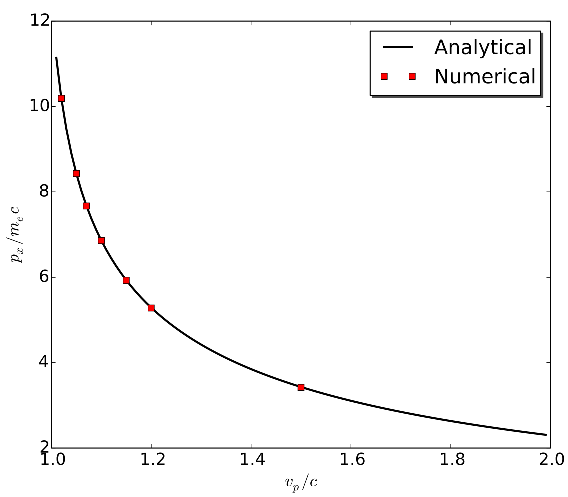

Figure 2. Peak longitudinal momentum p_x from a series of numerical calculations in which v_p is varied for $a_0 = 5$ against the predictions of Eq. (16).

4. Conclusions

We have reported on our recent work on electron motion in laser fields that is relevant to DLA of electrons in laser-plasma interactions. We have derived a solution for the classic problem of an electron in a plane wave for arbitrary phase velocity of the EM wave. EM waves in a plasma or waveguide structure can have superluminal phase velocities. Here we show the effect of this, and it is clear, as it should be from qualitative considerations, that this might be deleterious to DLA. The significance of this effect should be carefully considered in future studies of DLA.

5. Acknowledgements

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