Strong fields and slow light

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1 Introduction

The implications of the quantum vacuum have fascinated researchers over many decades, and contemporary developments in ultra-high intensity laser technology have recently reinvigorated interest in its electromagnetic properties. ELI [1] and its potential successors, such as IZEST [2], promise to offer laser fields of such high intensity that their interaction with the quantum vacuum will be manifest. Such projects present a new experimental paradigm for fundamental physics.

Hierarchies of scale are inherent in nature and, from a practical viewpoint, precise details of very high energy degrees of freedom are unimportant for modelling the behaviour of lower energy degrees of freedom. From the perspective of the path integral formulation of quantum theory, high energy degrees of freedom associated with the quantum vacuum may be "integrated out" to reveal an effective action for the electromagnetic field at lower energy scales. The remnant of the quantum vacuum is an electromagnetic self-coupling due to vacuum fluctuations, and the Euler-Lagrange equations induced from an effective action capture the behaviour of the electromagnetic field at intensities where further quantum effects do not noticeably intrude. The purpose of the present article is to highlight some recent mathematical observations involving non-linear waves that may be of interest to high-intensity laser scientists.

Perhaps the most celebrated effective action was first derived in QED by Heisenberg and Euler [3], and was later recovered by Schwinger using his proper time formalism [4]. The Euler-Heisenberg Lagrangian is the result of integrating out the electron-positron field while treating the electromagnetic field as a classical external source. On a more speculative note, one of the most famous effective Lagrangians in non-Standard Model particle physics, the Born-Infeld Lagrangian, was obtained by Fradkin and Tseytlin [5] from a bosonic open string path integral in an external electromagnetic field (although the historical origins of the Born-Infeld Lagrangian are considerably older than string theory, and even pre-date QED [6]).

In the absence of additional sources, the Euler-Lagrange equations associated with an effective action are the Maxwell equations

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla\cdot\mathbf{D} = 0, \qquad \nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} \tag{2}$$

where $\mathbf{D}, \mathbf{H}$ are *non-linear* functions of $\mathbf{E}, \mathbf{B}$ and their derivatives. Thus, in this formalism, the quantum vacuum presents itself as a non-linear polarizable medium and it follows that, in general, light will not propagate rectilinearly at speed c through an ambient electric or magnetic field.

An effective Lagrangian induced from a Lorentz invariant quantum theory will be Lorentz invariant. The Euler-Heisenberg and Born-Infeld Lagrangians are obtained when derivatives of $\mathbf{E}, \mathbf{B}$ are neglected, and the only local Lorentz invariants that can be formed from $\mathbf{E}$, $\mathbf{B}$ exclusively are X, Y where[1]

$$X = |\mathbf{E}|^2 - |\mathbf{B}|^2, \qquad Y = 2\mathbf{E}\cdot\mathbf{B}. \tag{3}$$

The effective Lagrangian $\mathcal{L}$ depends on the electromagnetic field via X, Y only, and the fields $\mathbf{D}$, $\mathbf{H}$ are deduced from $\mathcal{L}(X, Y)$ as follows:

$$\mathbf{D} = 2\frac{\partial\mathcal{L}}{\partial X}\mathbf{E} + 2\frac{\partial\mathcal{L}}{\partial Y}\mathbf{B}, \tag{4}$$

$$\mathbf{H} = 2\frac{\partial\mathcal{L}}{\partial X}\mathbf{B} - 2\frac{\partial\mathcal{L}}{\partial Y}\mathbf{E}. \tag{5}$$

The Lagrangian for classical vacuum Maxwell theory is

$$\mathcal{L}_{\mathrm{M}} = \frac{X}{2} \tag{6}$$

while the Born-Infeld and (weak field) Euler-Heisenberg Lagrangians have the form

$$\mathcal{L}_{\mathrm{BI}} = \frac{1}{\kappa^2}\left(1 - \sqrt{1 - \kappa^2 X - \frac{\kappa^4}{4}Y^2}\right) \tag{7}$$

and

$$\mathcal{L}_{\mathrm{EH}} = \frac{X}{2} + \frac{2\alpha^2}{45m^4}\left(X^2 + \frac{7}{4}Y^2\right) \tag{8}$$

respectively, where $\alpha = \frac{e^2}{4\pi} \sim 1/137$ is the fine structure constant, m is the rest mass of the electron and e is the elementary charge, whereas the value of the Born-Infeld constant κ is unknown.

[1] This article uses Heaviside-Lorentz units with $c = \hbar = 1$.

The Born-Infeld Lagrangian $\mathcal{L}_{\mathrm{BI}}$ has a privileged mathematical status. Its field equations have a number of interesting qualities that single out $\mathcal{L}_{\mathrm{BI}}$ from the family of Lagrangians of the form $\mathcal{L}(X, Y)$ [7, 8, 9]. Of particular interest here is the fact that, despite their highly non-linear structure, the Born-Infeld field equations have a number of closed-form exact solutions [10] including those modelling a plane wave in a constant background field [11].

2 Plane electromagnetic waves in regions of constant field

2.1 Constant magnetic field

Consider an electromagnetic plane wave travelling in the z-direction through a constant magnetic field $\mathbf{B}_0 = (B_{0x}, B_{0y}, B_{0z})$ given as

$$\mathbf{E} = \left(-v\mathcal{E}(z - vt),\ 0,\ \chi\mathcal{E}(z - vt)\right), \tag{9}$$

$$\mathbf{B} = \left(B_{0x},\ B_{0y} - \mathcal{E}(z - vt),\ B_{0z}\right). \tag{10}$$

Here v is the phase speed of the wave, the smooth function $\mathcal{E}$ of a single variable encodes the profiles of the electric and magnetic fields of the wave and χ is a real constant (to be specified). Remarkably, it can be shown [11] that the Born-Infeld field equations (i.e. those that arise from (1-5) with $\mathcal{L} = \mathcal{L}_{\mathrm{BI}}$) possess *exact* solutions of the form (9, 10) where $\mathcal{E}$ is arbitrary and

$$v^2 = \frac{1 + \kappa^2 B_{0z}^2}{1 + \kappa^2 (B_{0x}^2 + B_{0y}^2 + B_{0z}^2)}, \tag{11}$$

$$\chi = \frac{\kappa^2 B_{0x} B_{0z} v}{1 + \kappa^2 B_{0z}^2}. \tag{12}$$

Clearly, the background magnetic field $\mathbf{B}_0$ affects the phase speed v of the wave and $v \leq 1$. The background magnetic field slows the plane wave down.

However, the Born-Infeld field equations are not the only system that possesses exact solutions of the form (9, 10). If $B_{0y} = 0$ then the plane wave (9, 10) is an exact solution to (1-5) if the Lagrangian has the form

$$\mathcal{L}(X, Y) = c_1 + c_2 Y + \mathcal{F}(X + \lambda Y^2) \tag{13}$$

where c_1, c_2, λ are constants, $\mathcal{F}$ is an *arbitrary* smooth real function of a single variable and

$$v^2 = \frac{1 + 4\lambda B_{0z}^2}{1 + 4\lambda (B_{0x}^2 + B_{0z}^2)}, \tag{14}$$

$$\chi = \frac{4\lambda B_{0x} B_{0z} v}{1 + 4\lambda B_{0z}^2}. \tag{15}$$

The terms in (13) involving c_1, c_2 can be immediately verified by inspection of (1, 2, 4, 5) without appealing to (9, 10). However, the generic nature of the third term is somewhat startling (the details of the derivation of (13) are given in Ref. [12]). The Born-Infeld Lagrangian is revealed when $c_1 = c_2 = 0$ and $\mathcal{F}(\xi) = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 4\lambda\xi}\right)$ with $\lambda = \kappa^2/4$.

2.2 Constant electric field

It is natural to consider the effect of a background electric field $\mathbf{E}_0 = (E_{0x}, E_{0y}, E_{0z})$, instead of a magnetic field, on the plane wave. It can be shown [11] that the ansatz

$$\mathbf{E} = \left(E_{0x} - v\mathcal{E}(z - vt),\ E_{0y},\ E_{0z} + \chi\mathcal{E}(z - vt)\right), \tag{16}$$

$$\mathbf{B} = \left(0,\ -\mathcal{E}(z - vt),\ 0\right) \tag{17}$$

is an exact solution to the Born-Infeld field equations with

$$v^2 = 1 - \kappa^2 \left(E_{0x}^2 + E_{0y}^2\right), \tag{18}$$

$$\chi = \frac{\kappa^2 E_{0x} E_{0z}}{v}. \tag{19}$$

Moreover, if $E_{0x} = 0$ then an application of the methodology introduced in Ref. [12] reveals that (16, 17) is an exact solution to (1-5) if, once again, the Lagrangian has the generic form (13) and

$$v^2 = 1 - 4\lambda E_{0y}^2, \tag{20}$$

$$\chi = 0. \tag{21}$$

Interestingly, (20) is independent of the background field component E_{0z} (the component of the background electric field $\mathbf{E}_0$ in the direction of the propagation of the electromagnetic wave). However, the phase speed of the wave is not equal to 1 when the field component E_{0y} is non-zero.

3 QED vacuum polarization and slow light

It is straightforward to see that a simple application of the previous results immediately yields phase speeds of plane waves in the QED vacuum polarized by a background field. If $\mathcal{F}$ behaves as

$$\mathcal{F}(\xi) = \frac{1}{2}\xi + \frac{2\lambda}{7}\xi^2 + \mathcal{O}(\lambda^2) \tag{22}$$

then it follows

$$\mathcal{F}(X + \lambda Y^2) = \frac{1}{2}X + \lambda\frac{2}{7}\left(X^2 + \frac{7}{4}Y^2\right) + \mathcal{O}\left(\lambda^2\right) \tag{23}$$

and hence $\mathcal{L}_{\mathrm{EH}}(X, Y) = \mathcal{F}\left(X + \lambda Y^2\right) + \mathcal{O}(\lambda^2)$ where $\lambda = \frac{7\alpha^2}{45m^4}$. Thus, substituting $\lambda = \frac{7\alpha^2}{45m^4}$ in (14, 20) and expanding the phase speeds to first order in α^2 immediately gives

$$v = 1 - \frac{14\alpha^2}{45m^4} B_{0x}^2 + \mathcal{O}\left(\alpha^4\right) \tag{24}$$

and

$$v = 1 - \frac{14\alpha^2}{45m^4} E_{0y}^2 + \mathcal{O}\left(\alpha^4\right) \tag{25}$$

respectively, which are the phase speeds of polarization modes corresponding to (9, 10) and (16, 17). The above is a novel derivation of well-known results previously obtained by linearizing the Euler-Heisenberg field equations [13].

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