Effect of Landau quantization on the equations of state in dense plasmas with strong magnetic fields

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Introduction

The equations of state (EOS) are the fundamental relation between the macroscopically quantities describing a physical system in equilibrium[1]. The EOS relates all thermodynamic quantities, such as density, pressure, energy, entropy, etc. Knowledge of the EOS is required in order to solve hydrodynamic equations in specific physical situations, such as plasma physics associated with laser interaction with matter, shock wave physics, astrophysical objects etc. The properties of matter are summarised in the EOS.

The concept of Landau quantization in the presence of strong magnetic fields is presented in the rest of this section and, in the next section, the electron EOS in the presence of strong magnetic fields is calculated and presented for non-relativistic plasmas for both zero and finite temperatures. Finally, the regime of applicability to laser-plasma interactions is discussed. The short space in this report prevents any detailed description of these concepts – the interested reader is directed to our recent paper where Landau quantisation is discussed in depth[1].

These EOS were previously scattered throughout the astrophysical literature and were caste in terms of densities of 10^6gcm^{-3} and magnetic fields $>10^{12}$G, which are, of course, appropriate for those expected in neutron star atmospheres. The role of this report is to reformulate the EOS in terms of those that can potentially be realised in laboratory plasmas. By doing so, we have two aims in mind: first, to alert the experimental laser-plasma physics community to the potential of realising Landau quantisation in the laboratory for the first time since the theory was first formulated; second, to indicate that there are significant differences in the EOS when these strong magnetic fields are present, as discussed in some detail our recent paper for the interested reader.

If one neglects the electron-phonon interactions then the thermodynamic functions can be expressed as a superposition of the electrons and the atoms (or ions) of the medium under consideration. For example, one can write the energy E and the pressure P in the following form

$$E = E_e + E_i \qquad P = P_e + P_i \tag{1}$$

where E_e and P_e are the electron energy and pressure accordingly while E_i and P_i are the contributions of the atom (or ion in a plasma medium) vibrations to the energy and pressure. For example, the Debye EOS for ions is given by[2]

$$E_i = 3Nk_B T D\left(\frac{T_D}{T}\right) \qquad P_i = \frac{3\Gamma N k_B T}{V} D\left(\frac{T_D}{T}\right) \qquad D(x) \equiv \frac{3}{x^3}\int_0^x \frac{y^3 dy}{e^y - 1} \tag{2}$$

where Γ is phenomenological coefficient and T_D is the Debye temperature. A very useful phenomenological EOS for a solid is given by the Gruneisen EOS[2],

$$P_i = \frac{\gamma V E_i}{V} \qquad \gamma = \frac{3\alpha V}{\kappa c_V} \tag{3}$$

where the quantities on the right hand side of the second equation can be measured experimentally: α = linear expansion coefficient, κ = isothermal compressibility, c_V = specific heat at constant volume. There are more sophisticated EOS for ions[3),4)], however in this report we do not consider further the ion contributions. Since the magnetic field affects mainly the electrons, the electronic contribution is discussed in this report.

Magnetic fields of a few GG appear to be within reach of the existing petawatt lasers[5]. In this report, the EOS are calculated in the presence of these magnetic fields. Since only the electrons are influenced by the magnetic field under consideration, only the electronic equation of state is reconsidered and analysed. For the ion part of the EOS one can take the Gruneisen EOS or any other appropriate model.

Landau Levels

The motion of a charged particle in a magnetic field is quantized[6]. In particular, the motion of an electron (mass m and charge –e,) perpendicular to a constant magnetic field (B) is quantized (the vectors are denoted by bold letters). The kinetic energy of the transverse motion is quantized into Landau levels with a quantum number n_L. The Landau energy levels are defined by[7-9]

$$\varepsilon_\perp = \frac{1}{2}m\mathbf{v}_\perp^2 = \frac{\mathbf{\Pi}_\perp^2}{2m} = \left(n'_L + \frac{1}{2}\right)\hbar\omega_B\ ;\ n'_L = 0,1,2,3,...$$
$$\mathbf{\Pi} = \mathbf{p} + \frac{e\mathbf{A}}{c} = m\mathbf{v} = m\frac{d\mathbf{r}}{dt} \tag{4}$$
$$\mathbf{p} = -i\hbar\nabla\ ;\ \mathbf{B} = \nabla\times\mathbf{A}$$
$$\omega_B = \frac{eB}{mc}$$

$\hbar = h/2\pi$, h is the Planck constant, r and v are the electron position in space and velocity accordingly, Π is the mechanical momentum, p is the canonical momentum and A is the vector potential. The classical equation of motion for an electron at a position r and its solution in the perpendicular direction are given by

$$\frac{d\Pi}{dt} = \left(\frac{-e}{c}\right)\left(\frac{d\mathbf{r}}{dt}\right)\times\mathbf{B} \tag{5}$$

$$\left|\mathbf{r}_\perp - \mathbf{R}_c\right| = \frac{c\Pi_\perp}{eB} = r_B$$

$$r_B = \frac{mcv_\perp}{eB} = \left(2n'_L + 1\right)^{1/2}\left(\frac{\hbar c}{eB}\right)^{1/2} \tag{6}$$

$$\mathbf{R}_c = \mathbf{r}_\perp + \left(\frac{c}{eB^2}\right)\mathbf{B}\times\mathbf{\Pi}_\perp$$

R_c is the position vector of the guiding centre of the electron gyromotion, while the Landau quantization (4) has been used in deriving the quantized radius of gyration r_B.

The energy of the electron (E) is given by

$$\varepsilon = n_L\hbar\omega_B + \frac{p_z{}^2}{2m} \tag{7}$$

$$n_L = n'_L + 1/2 + 1/2\, s\ ;\ \ n_L = 0,1,2...;n'_L = 0,1,2...;s = \pm 1.$$

The spin energy ε_s of the electron is included in the energy equation

$$\varepsilon_S = \frac{1}{2}\hbar\omega_B s\ ;\ \ s = \pm 1 \tag{8}$$

For the ground level ($n'_L = 0$) s = -1 the degeneracy is 1, for exited levels s = -1 or =1 and therefore the degeneracy is 2. For extremely large magnetic fields

$$\hbar\omega_B \geq mc^2 \quad \Rightarrow \quad B \geq B_{rel} = \frac{m^2c^3}{e\hbar} = 4.41\cdot 10^{13}\ [G] \tag{9}$$

the transverse motion of the electron becomes relativistic, and in this case the following energy of the electron is obtained from the Dirac equation (10)

$$\varepsilon = \left(m^2c^4 + p_z{}^2c^2 + 2n_L mc^2\hbar\omega_B\right)^{1/2} = \left[p_z{}^2c^2 + m^2c^4\left(1 + \frac{2n_L B}{B_{rel}}\right)\right]^{1/2} \tag{10}$$

The relativistic motion of the electrons [Equation (10)] is strongly influenced by the extremely high magnetic fields (B_{rel}). For the laser plasma interactions we shall define the high magnetic fields by

$$\hbar\omega_B \geq \frac{e^2}{a_0} \quad \Rightarrow \quad B \geq B_0 = \frac{m^2e^3c}{\hbar^3} \approx 2.35\cdot 10^9\ [G]$$
$$a_0 = \frac{\hbar^2}{me^2} \approx 5.29\cdot 10^{-9}\ [cm] \tag{11}$$

where a_0 is the Bohr radius. For B much larger than B_0, the electron cyclotron energy $\hbar\omega_B$ is larger than the typical Coulomb energy, and therefore the usual perturbation in laser plasma interaction of the magnetic field effects relative to the Coulomb interactions is not permitted anymore. In this case the Coulomb forces act as a perturbation to the magnetic forces. The dominant length dimension for the very high magnetic fields is the first Landau radius r_B ($n_L = 0$) rather than the Bohr radius. It is interesting to point out that B_{rel} is related to B_0 by the fine structure constant α,

$$B_0 = \alpha^2 B_{rel}$$
$$\alpha = \frac{e^2}{\hbar c} \tag{12}$$

EOS in Strong Magnetic Fields, the Non-Relativistic Case

In this section we consider the non-relativistic EOS of a free electron gas in strong magnetic field at finite temperature[7-9]. As is described in introduction, the electron motion perpendicular to the magnetic field is determined by the magnetic field and the electron energy levels are quantized into Landau states. The energy spectrum of one electron is given by

$$\varepsilon = \varepsilon_\perp + \varepsilon_z = n_L\hbar\omega_B + \frac{p_z{}^2}{2m} \tag{13}$$
$$n_L = n'_L + 1/2 + s/2 = 0,1,2,...;n'_L = 0,1,2...s = \pm 1.$$

p_z is the electron momentum along the magnetic field and is continuous.

A most important difference between zero and non-zero (or rather small or large) magnetic field is "the counting of the number of states". In particular,

$$\textbf{number of states in phase space } (\mathbf{B} = 0) = \frac{\Delta x\Delta y\Delta z\Delta p_x\Delta p_y\Delta p_z}{h^3} = \frac{V\Delta p_x\Delta p_y\Delta p_z}{h^3} \tag{14}$$

$$\textbf{number of states in phase space } [\mathbf{B} = (0,0,B)] = \frac{V\Delta p_x\Delta p_y\Delta p_z}{h^3} = \frac{V\left(2\pi p_\perp\Delta p_\perp\right)\Delta p_z}{h^3} = \frac{\left(2\pi m\Delta\varepsilon_\perp\right)\Delta p_z}{h^3} = \frac{VeB\,\Delta p_z}{h^2c}$$

This degeneracy has to be multiplied by the quantum degeneracy $g_{nL} = 1$ for $n_L = 0$ and $g_{nL} = 2$ for $n_L = 1,2,3\ldots$. In calculating the number of states, with a magnetic field in the z direction, the Landau quantization of Equation (14) has been used.

The starting point in calculating the EOS without or with magnetic field is the grand partition function for Fermi-Dirac particles (in this case, an electron gas). However, in the case of a large B one has to use the electron energy given in (13), and when changing the sums with integrals, starting with the grand partition function, we obtain in this case the following equations of state (the details of which can be found in reference [1)]):

$$N = nV = \frac{VeB(mk_BT)^{1/2}\sqrt{2}}{h^2c}\sum_{n_L=0}^{\infty} g_{nL}F_{-1/2}\left(\frac{\mu - n_L\hbar\omega_B}{k_BT}\right) \equiv \sum_{n_L=0}^{\infty} N_{n_L}$$

$$P = \frac{eB(k_BT)^{3/2}m^{1/2}2\sqrt{2}}{h^2c}\sum_{n_L=0}^{\infty} g_{nL}F_{1/2}\left(\frac{\mu - n_L\hbar\omega_B}{k_BT}\right) \tag{15}$$

$$E = \frac{PV}{2} + \hbar\omega_B\sum_{n_L=0}^{\infty} n_L N_{n_L}$$

$$S = \frac{E + PV - \mu N}{T} = \frac{3PV}{2} + \hbar\omega_B\sum_{n_L=0}^{\infty} n_L N_{n_L} - \frac{\mu N}{T}$$

F_k is the Fermi-Dirac integral of the order k. From the first of the Equations in (15), μ is calculated as a function of n, and this value is used in the other equations to calculate the other thermodynamic quantities (P, E, S) as a function of n (or density) and T for a given magnetic field B.

It is interesting to point out that the Fermi-Dirac integrals for the electron density n and the electron pressure P are different (one order less) than in the B = 0 case. This is a result of the difference in the number of states for small or large B. If the magnetic field is very large then only the ground state is occupied, and in this case only the first term contributes to the infinite sum in (14) ($n_L = 0$, $g_{nL} = 1$).

For small temperatures, i.e. strong electron degeneracy, defined by

$$k_B T << \varepsilon_F - n_L \hbar\omega_B \ ; \quad \varepsilon_F \equiv \mu(T=0) \qquad (16)$$

one can use the expansion of the Fermi-Dirac integrals in order to write explicitly the temperature contribution to the EOS. The EOS, given in the general case by (15), can be expressed in the lowest order in temperature by the following equations;

$$n = \frac{2eB\sqrt{2m}}{h^2 c}\mu^{1/2}\left\{\left[1+2\sum_{n_L=1}^{n_{max}}\left(1-\frac{n_L\hbar\omega_B}{\mu}\right)^{1/2}\right] - \frac{\pi^2}{24}\left(\frac{k_B T}{\mu}\right)^2\left[1+2\sum_{n_L=1}^{n_{max}}\left(1-\frac{n_L\hbar\omega_B}{\mu}\right)^{-1/2}\right]\right\}$$

$$P = \frac{4eB\sqrt{2m}}{h^2 c}\mu^{3/2}\left\{\left[1+2\sum_{n_L=1}^{n_{max}}\left(1-\frac{n_L\hbar\omega_B}{\mu}\right)^{3/2}\right] + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\left[1+2\sum_{n_L=1}^{n_{max}}\left(1-\frac{n_L\hbar\omega_B}{\mu}\right)^{-1/2}\right]\right\}$$

$$S = \frac{\pi^2 k_B{}^2 eB\sqrt{2m}}{h^2 c}\mu^{-1/2}\left[1+2\sum_{n_L=1}^{n_{max}}\left(1-\frac{n_L\hbar\omega_B}{\mu}\right)^{-1/2}\right]T$$

$$E = -PV + TS + \mu n V \qquad (17)$$

We suggest the use of Equation (17) as the EOS input in laser plasma electron transport codes for the phase space domain where Landau quantization is relevant. It is important to realize that for laser-solid target interactions the changes in the EOS induced by Landau quantization can be significant. For strong Landau quantization, i.e. $n_L = 0$, the Fermi energy ε_F, the pressure P (denoted here by P_B) and the speed of sound c_B scale with respect to the electron density n and the magnetic field B in the following way,

$$\varepsilon_F \propto \left(\frac{n^2}{b^2}\right), P_B \propto B\varepsilon_F{}^{3/2} \propto \frac{n^3}{B^2}, c_B \propto \left(\frac{\partial P}{\partial \rho}\right)^{1/2} \propto \frac{n^2}{B^2} \qquad (18)$$

These scaling laws are significantly different from those derived from Fermi Dirac EOS at B=0. We find that

$$P_B(MB) \approx 0.69\left(\frac{n_{23}{}^3}{B_9{}^2}\right)$$
$$P_0(MB) \approx 0.5(n_{23})^{5/3} \qquad (19)$$
$$B_9 \equiv \frac{B}{10^9 G}$$

It is important to emphasize that these relations are only valid for $n_{23} < 1.3\ B_9^{1.5}$ and for $T < T_B \sim 11.58\ B_9$.

The most interesting and important question is the identification of the plasma domain (density and temperature) with the high magnetic field created in laser-plasma interactions. Conservation of energy requires that the magnetic field be highly localized and will be generated in the density regime between critical and solid (or higher in the case of shock compressed plasmas).

The crucial factor appears to be the temperature profile at this point. The study of the background electron temperature under these conditions is a topic of active study. Modeling of these conditions relies upon the accurate calculation of the electric and magnetic fields associated with the fast electron propagation inside the solid target and the return current that is required to compensate for the multi-TAcm^{-2} beam. The most modern tools rely upon relativistic Vlasov-Fokker-Planck modeling of the fast electron transport and return current, principally because of the long mean-free paths of the fast particles involved. They require rigorous testing against experiment and this is an ongoing investigation.

In summary, magnetic fields of 0.7GG have already been observed with a 100TW glass laser system, and there are realistic possibilities that multi-GG B-fields can be generated with existing PW-class laser systems. The B-fields were measured with oblique incidence p-polarized laser irradiation. Particle-in-cell modeling suggests that the azimuthal B-field lies outside the main interaction region in the colder regions of the target. It appears that for a few cycle petawatt laser pulses with ten-femtosecond pulse duration, a domain might be obtained where the Landau quantization plays an important role in the EOS data and also be a dominant factor in determining the transport coefficients.

The role of this review paper has been to alert the experimental laser-plasma physics community to these exciting but challenging requirements for significant changes in the EOS. If they can be realized in the laboratory, they will mark a significant breakthrough for high energy density plasma science.

References

1. S Eliezer, P A Norreys, J T.Mendonca and K L Lancaster, Phys. Plasmas 12, 052115 (2005).
2. S Eliezer, A Ghatak, H Hora with an introduction by Teller E. Fundamentals of Equations of State World Scientific, New Jersey (2002).
3. D J Steinberg, Equation of State and Strength Properties of Selected Materials, Lawrence Livermore National Laboratory report UCRL-MA-106439 (1991).
4. AV Bushman, G I Kanel, A L Ni and V E Fortov, Intense Dynamic Loading of Condensed Matter, Taylor and Francis, Washington (1993).
5. M Tatarakis, A Gopal *et al.*, Phys. Plasmas 9, 2244 (2002).
6. L D Landau and E M Lifshitz, Quantum Mechanics 3rd ed. Pergamon, Oxford, UK (1977).
7. D Lai, Rev. Mod. Phys. 73, 629 (2001).
8. D Lai and S L Shapiro, Ap. J. 383, 745 (1991).
9. A M Abrahams and S L Shapiro, Ap. J. 374, 652 (1991).