Wave breaking limits for relativistic electrostatic waves in a warm plasma

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Introduction

It was shown by Dawson [1] that wave breaking in a cold plasma occurs when elements of the plasma electron fluid that started out in different positions overtake each other while moving back and forth during the passage of the wave. For both non-relativistic and relativistic plasmas, this overtaking happens when the peak fluid velocity equals the phase speed of the plasma wave [1,2]. A direct consequence of this is that a large fraction of the plasma electrons get trapped in and accelerated by the plasma wave.

Adapting this definition for use with warm plasmas is by no means straightforward. Although it is generally accepted that wave breaking implies the trapping of background plasma electrons [3], it has been observed by several authors that electron trapping in itself does not imply wave breaking [4,5]. The reason for this is that in a thermal plasma there is always a small fraction of the electron population attaining speeds larger than the phase speed of the plasma wave under consideration. Such electrons will of course get trapped sooner or later, so if wave breaking were equated to background electron trapping, every longitudinal plasma wave, regardless of phase speed or amplitude, would always be broken.

In practice, plasma waves can support small numbers of trapped electrons without losing their wave structure. These trapped particles may cause the wave to be Landau-damped, but as long as the wave structure remains intact, the wave cannot be considered broken. This is well visualized, for example, by Bergmann and Mulser [5], as well as in simulations [6] and experiments [7] of self-injected mono-energetic electron bunches in the "bubble regime". In their paper, Bergmann and Mulser present the results of Vlasov simulations showing that when a small fraction of the plasma electrons gets trapped, the periodic wave structure is unaffected, while in the case that a large fraction gets trapped, the periodic structure collapses and disappears. In this context, wave breaking is defined as "the loss of periodicity in at least one of the macroscopically observable quantities." In terms of particle trapping, this condition is satisfied if a considerable fraction of the plasma electrons are trapped, not just the fastest ones.

In order to quantify this definition, the electron sound speed $s_0 = (3kT_0/m_e)^{1/2}$ and associated momentum $p_0 = (3kT_0m_e)^{1/2}$ are introduced, where T_0 denotes the plasma temperature before the arrival of the wave. Since Langmuir waves in an electron plasma always have a phase speed larger than s_0, it follows that electrons with initial speed $|v| \le s_0$ will contribute fully to the collective oscillations that define the wave, while electrons with initial speed $|v| > s_0$ will always be faster than some of the Fourier components of the Langmuir wave and will thus not contribute fully to its oscillations. Wave breaking is then defined as *the trapping (by the wave) of background plasma electrons having an initial forward momentum not larger than p_0*. In other words, a wave will break if not only electrons from the "tail" of the distribution are trapped, but also electrons originating from the "body" of the distribution. When that happens, the wave will disrupt the collective electron oscillations that drive it in the first place, and will eventually collapse and lose its structure.

In practice, however, a number of less straightforward and often conflicting definitions for wave breaking in a warm plasma are used. All these definitions are based on the 1-D Vlasov equation for electrons, to which the quasi-static approximation is applied, i.e. every quantity is assumed to be a function of $\xi = x - v_\varphi t$ only, where v_φ denotes the phase speed of the wave. Any solutions to the quasi-static Vlasov equation will break down as soon as the quasi-static assumption is violated, i.e. as soon as large-scale particle trapping starts. In that light it makes sense to equate the breakdown of the quasi-static Vlasov model to wave breaking. Unfortunately, most papers on wave breaking simplify the Vlasov model to a quasi-static warm-fluid model, and equate wave breaking to breakdown of that fluid model, sometimes referred to as *Coffey's criterion*. This is where things go wrong: many different warm-fluid models can be found in literature [3,8-14], but quite a few of them don't break down at the same instant that the Vlasov model breaks down, leading to a range of conflicting "definitions" for wave breaking. As will be shown below, this is particularly the case for ultra-relativistic plasma waves, i.e. plasma waves obeying $\gamma_\varphi^2\beta$, >> 1, where γ_φ is the Lorentz factor corresponding to the phase speed of the plasma wave, $\beta = (3kT_0/mc^2)$, T_0 the initial plasma temperature and m the electron mass.

It is demonstrated in this paper that great care must be taken in verifying that the breakdown of any fluid model used for the study of wave breaking coincides with the breakdown of the original quasi-static Vlasov model, and that a correct relativistic warm-fluid model is used that is also valid in the ultra-relativistic regime. It is shown here that for models meeting these two criteria [8] higher, more realistic wave breaking limits are found than for models that fail to meet them [10-14]. In an extension to the model of Katsouleas and Mori [8], a lower bound to the wave breaking field is derived here for the first time, as a complement to the upper bound presented there. Using this new lower bound, it is shown that a wave having phase speed $v_\varphi = c$ in a plasma with a finite initial temperature will not ever break, regardless of its amplitude.

This result has important repercussions for the research into multi-GeV electron acceleration in laser-wakefields [6,15]. Very large electric fields are needed for this acceleration scheme, so it would be problematic if there would exist a finite upper bound for the wave-breaking limit in a warm plasma even for $v_\varphi \to c$, as predicted by several authors [10,12-14]. Fortunately, it is found here that the wave-breaking limits calculated in those papers are inherently too low, and that the wave breaking threshold in the limit $v_\varphi \rightarrow c$ tends to infinity, as derived by Katsouleas and Mori [8].

Relativistic fluid dynamics

Before embarking on the study of wave breaking, we will give a short summary of relativistic fluid dynamics [16-18], since there are important differences with non-relativistic fluid dynamics. We will also clarify the differences between fully relativistic fluid dynamics and the so-called *warm-plasma approximation* [19], which is used when the mean plasma velocity is relativistic, but the thermal velocity spread is not.

In non-relativistic fluid dynamics, the internal energy U is a second order moment of the velocity distribution which can be separated exactly into the mean-flow energy and the thermal energy. The energy flux is a third order moment which can be separated exactly into the flux of mean energy, the flux of thermal energy along the mean flow, and the flux of thermal energy from one fluid element to the next. The latter quantity is often denoted as the heat flow. The fluid pressure P is a second order moment satisfying P = 2U for a 1-D fluid, or P = 2U/3 for a 3-D fluid. The ratio of heat capacities is given by $c_p/c_v = (U+P)/U$, so $c_p/c_v = 3$ for a 1-D fluid, and $c_p/c_v = 5/3$ for a 3-D fluid. On adiabatic compression, i.e. when the heat flow during compression is negligible, we have $P \sim n^{cp/cv}$, or $P \sim n^3$ for a 1-D fluid.

In relativistic fluid dynamics [16-18], things are quite different however. The internal energy is still a second order moment, but it cannot be separated directly into mean and thermal energy any more, since the rules for adding velocities have changed. The relativistic energy flux is a second order moment of the relativistic momentum distribution rather than a third order moment, and once again it is not straightforward to separate it into mean energy transport, thermal energy transport, and heat flow. The fluid pressure P is still a second order moment, but now satisfies the equation $U \geq P/2 + [1+ (P/2)^2]^{1/2}-1$. This means that c_p/c_v ranges from 3 for a non-relativistic plasma ($P << 1$) to 2 for an ultra-relativistic plasma ($P >> 1$).

An approximation that is used quite often in the theory of relativistic plasma dynamics is the so-called warm-plasma approximation [19]. In this approximation, it is assumed that the plasma temperature is much smaller than the kinetic energy associated with the mean flow. Among other things, this allows the approximate (not exact) separation of the internal energy and energy flux into contributions comparable to the non-relativistic case. The heat flow can then be approximated by third-order centered moments (centered with respect to the mean momentum), and the pressure can be written as $P \sim n^3$ for not-too-extreme adiabatic compression. As such, the warm-plasma approximation is more like non-relativistic fluid dynamics, avoiding most of the intricacies of the fully relativistic theory. As we shall see below, this approximation is not suitable for the study of breaking of (ultra-)relativistic plasma waves, since such waves may drive the plasma to temperatures for which the warm-plasma approximation is no longer applicable.

A special note of warning should be issued concerning the study of adiabatic processes in the plasma, since fast wave propagation is normally considered to be adiabatic. For non-relativistic plasmas, and in the warm-plasma approximation [19], the heat flow is given by third-order moments of the momentum distribution. Thus, for an adiabatic process, the third-order moments will vanish, providing a neat way to close the system of moment equations and arrive at a warm-fluid model. For a fully relativistic plasma however, the heat flow is not given by the third-order moments any longer [16-18], and setting the third-order moment to zero is no longer equivalent to considering an adiabatic process. In fact, the third-order moments can grow fairly large for ultra-relativistic adiabatic processes, and uncompromisingly forcing them to zero will lead to incorrect results, possibly even violation of the first and second law of thermodynamics.

Wave breaking and particle trapping

As stated above in our definition of wave breaking, wave breaking and background particle trapping are intimately related, and many attempts were made in the past to unite them. However, such attempts were often complicated by the fact that most models for plasma wave propagation employ the quasi-static approximation, which does not tolerate any particle trapping at all: as soon as the first particle gets trapped, any quasi-static model will break down immediately. The solution to this is to employ a quasi-static model based on a so-called waterbag distribution: a distribution that is a non-zero constant for thermal speeds smaller than the electron sound speed, and zero otherwise. This distribution behaves like a Gaussian distribution in many aspects, but particle trapping will be postponed until particles at (initially) the electron sound speed can be trapped. Thus, a quasi-static waterbag model will break down at the same instant that large-scale particle trapping sets in for a non-quasi-static Gaussian model, triggering wave breaking according to our definition. This renders the quasi-static waterbag model quite suitable for wave breaking, since the point of breakdown of this model can be determined analytically to great accuracy.

The quasi-static waterbag model is employed explicitly by Coffey [3] and Katsouleas and Mori [8], who define wave breaking by the trapping of the upper waterbag boundary. It should be noted however, that most other work on wave breaking still favours the waterbag distribution implicitly, even if it is not mentioned. This is because almost all work on plasma wave breaking starts from the following assumptions: (i) the wave is quasi-static, and (ii) its propagation is an adiabatic process. As it happens, the waterbag distribution is the only one that satisfies both assumptions exactly, so it is implicitly favoured over other distributions, which can only satisfy the assumptions approximately. So even papers that argue that they do not restrict themselves to a particular distribution function are still doing so, and are thus less general than they claim to be.

We have studied particle trapping for a wave on the verge of breaking for two different models: the fully relativistic model of Katsouleas and Mori [8], and the weakly relativistic model of Rosenzweig, Sheng and Schroeder [10,12-14]. For this, we used the Hamiltonian approach of Ruth and Chao for particle dynamics in a wakefield [20], where the wakefield potential was of course taken from the various plasma models we considered. It should be noted that for proper application of this method, a few common mistakes, made by several authors [10,14], should be avoided:

(i) one cannot express a particle's velocity directly as the sum of a mean velocity and a thermal part, since addition of relativistic velocities is governed by special rules, (ii) even though several authors do this, describing a warm plasma as a distribution of test electrons governed by the wakefield potential of a cold plasma is not correct, as it will miss out the effect of pressure on the relationship between mean velocity and electrostatic fields, (iii) the average speed of the electron fluid is *not* equal to the speed of a particle that is initially at rest; in fact, the mean fluid flow does not correspond to any individual particle orbit, and (iv) when the wakefield potential for a warm plasma is used, one must not add an additional term for the plasma pressure to the Hamiltonian, as the plasma pressure is a fictitious force acting on the mean plasma speed, and not a real force acting on individual particles.

When comparing the models of Katsouleas and Mori [8] and Rosenzweig [10,12-14] in the ultra-relativistic regime, we found that a wave on the verge of breaking according to the former model can trap particles having the electron sound speed initially. This means that the breakdown of this model corresponds to true wave breaking, i.e. loss of wave structure in the Vlasov solution. However, a wave on the verge of breaking according to Rosenzweig's model can only trap particles moving at twice the electron sound speed or faster; this means that Rosenzweig's model already breaks down long before any breakdown or loss of periodicity can be observed in the corresponding Vlasov solution. Thus, the "wave-breaking limits" as predicted by Rosenzweig's model are systematically too low, which should not be surprising given that their expression for the plasma pressure ($P \sim n^3$) is systematically too high for relativistic plasmas.

The systematic comparison of limits for wave breaking and particle trapping has revealed problems with a number of available models for warm-plasma wave breaking. A second attempt by Rosenzweig [11] to calculate a wave-breaking limit using a three-fluid model also exhibits shortcomings. While the model has been constructed in such a way that it breaks down when the fastest of three cold fluids gets trapped by a passing plasma wave, its peculiar velocity distribution, as well as the strange way in which closure of the system of moment equations has been obtained, lead to an effective pressure $P \sim n^{1/2}$, which is of course much too low. The resulting wave breaking limit is much too high, $E_{wb} \sim \gamma_\varphi^{1/2}$, resembling the limit for a cold plasma [2].

Another method that has been employed by several authors [21] is the so-called method of characteristics, which can be used to obtain solutions to the quasi-static Vlasov equation for arbitrary initial velocity distributions, i.e. not just waterbags. Integration of these solutions is carried out using the method of steepest descent. Because of certain approximations used in this approach, the solutions can tolerate some degree of particle trapping without breaking down, rendering this method ideal to study the relation between wave breaking and particle trapping. Unfortunately, both Aleshin and Khachatryan [21] carry this method too far, continuing to increase the wave amplitude until virtually all plasma particles have been trapped, and there are none left to drive the collective oscillations that make up the wave. In other words, the breakdown of such models occurs too late, long after the wave has already collapsed and disappeared. As a result, both the derived wave breaking limits and the alleged soliton formation for waves on the verge of breaking that have been obtained through this method appear to be the result of incorrect modelling rather than new physics. So while the method of characteristics can be a valuable tool in the study of wave breaking, and particle trapping analysis reveals that it must be used with care, and not pushed beyond its inherent limitations.

Wave breaking limit for relativistic waves in a warm plasma

In this section, we will present upper and lower bounds for the wave-breaking limit on the electric field of a plasma wave using the model of Katsouleas and Mori [8]. We use this model because our previous analysis has revealed that this is the only model that applies relativistic fluid dynamics correctly, while we have shown that other models yield results that are systematically too low or too high. Wave breaking can be defined as either the trapping of electrons having the electron sound speed initially, or the breakdown of the quasi-static model, as these definitions coincide for this model; see the previous section.

In their original paper, Katsouleas and Mori only derived an upper bound for the wave-breaking limit in the ultra-relativistic regime. However, in order to set it apart from Rosenzweig's approach [10], and in order to make the distinction between the ultra-relativistic regime (in which relativistic thermal effects are dominant) and the regime of today's laser-wakefield experiments (in which thermal effects form a small correction to the cold relativistic model of Akhiezer and Polovin [2]), we will derive lower and upper bounds for both the laser-wakefield regime and the ultra-relativistic regime.

In the laser-wakefield regime, $\gamma_\varphi^2\beta << 1$, the following lower and upper bounds for the wave-breaking field have been found:

$$E_{wb}^2 \geq 2\left[\gamma_\varphi - 1 - \gamma_\varphi\left(8\gamma_\varphi^{1/2}\beta^{1/4}/3 - 2\gamma_\varphi\beta^{1/2}\right)\right]$$
$$E_{wb}^2 \leq 2\left[\gamma_\varphi - 1 - \gamma_\varphi\left(\gamma_\varphi^{1/2}\beta^{1/4} - \gamma_\varphi\beta^{1/2}\right)\right]$$

These results can readily be recognized as the cold relativistic limit derived by Akhiezer and Polovin [2], with small thermal corrections. The bounds differ by a small amount only, and the lower bound is actually equal to the expression derived by Schroeder using Rosenzweig's model [10,13,14]. This follows from the fact that in the laser-wakefield regime the differences between Katsouleas and Mori's model and Rosenzweig's are actually quite small; however, it already goes to show that the values predicted by Rosenzweig's model are too low, as they provide only a lower bound for the true wave-breaking field.

In the ultra-relativistic regime, $\gamma_\varphi^2\beta$, >> 1, the following lower and upper bounds for the wave-breaking field have been found:

$$E_{wb}^2 \geq 2\left[\ln\left(\gamma_\varphi^{1/2}\beta^{1/4}\right)/2\beta^{1/2}\right]$$
$$E_{wb}^2 \leq 2\left[\ln\left(2\gamma_\varphi^{1/2}\beta^{1/4}\right)/2\beta^{1/2}\right]$$

The upper limit is a confirmation of the value found by Katsouleas and Mori, while the lower limit has been added to get a better grip on the behaviour of E_{wb} as γ_φ approaches infinity. It can be seen that both bounds approach infinity as γ_φ tends to infinity, taking the actual wave breaking limit with them. This emphasizes the difference with Rosenzweig's model, in which the wave-breaking limit tends to a finite value when γ_φ tends to infinity.

The behaviour of E_{wb} for large γ_φ, i.e. v_φ tends to c, has important consequences. In Katsouleas and Mori's model, a wave having finite amplitude and $v_\varphi = c$ cannot ever break, while in Rosenzweig's model, such a wave can still break provided its (finite) amplitude is large enough. However, a plasma wave cannot be considered broken if all plasma particles are slower than the wave, so breaking of a plasma wave having $v_\varphi= c$ implies that at least some plasma particles are accelerated to speed c. Since such particles would have infinite energy, they cannot be produced by a finite wave, so it follows that a finite wave cannot break at all. Katsouleas and Mori's model agrees with this, while Rosenzweig's model clearly fails on this count.

Conclusions

In conclusion, wave breaking of longitudinal waves in a warm plasma has been studied. A quantitative definition of wave breaking has been provided and compared against other definitions used in the literature. In a number of cases, wave breaking has been equated to breakdown of the mathematical model, without adequate verification that this coincides with a physical breakdown of the wave. Having studied the fluid dynamics for a warm, relativistic plasma it has been demonstrated that there is only one model on wave breaking that handles this correctly [8]. This model has been expanded here to derive both upper and lower limits for the electric field amplitude E_{wb} at wave breaking, for both $\gamma_\varphi{}^2\beta << 1$ and $\gamma_\varphi{}^2\beta >> 1$. It has been shown that in the latter regime $E_{wb}{}^2 \sim \ln(\gamma_\varphi{}^{1/2}\beta^{1/4}) /\beta^{1/2}$, which implies that for fixed β and $\gamma_\varphi \to \infty$, E_{wb} tends to infinity, so a wave with phase speed $v_\varphi = c$ will not ever break, regardless of its amplitude. This is particularly relevant for multi-GeV electron acceleration schemes [15], since such schemes rely on the possibility of generating very large longitudinal fields in plasmas. Finally a number of existing misunderstandings on the behaviour of a relativistic plasma wave near breaking has been clarified.

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