Electrostatic wave-breaking in thermal plasmas

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Introduction

High-power lasers and plasmas may be used to accelerate electrons by electric fields that are orders of magnitude greater than those achievable using conventional methods [1]. An intense laser pulse is used to drive a wave in an underdense plasma and, for sufficiently large fields, non-linearities lead to collapse of the wave structure ('wave-breaking') due to sufficiently large numbers of electrons becoming trapped in the wave.

Hydrodynamic investigations of wave-breaking were first undertaken for cold plasmas [2,3] and thermal effects were later included in non-relativistic [4] and relativistic contexts [5–7] (see [8] for a discussion of the numerous approaches). However, it is clear that the value of the electric field at which the wave breaks (the electric field's 'wave-breaking limit') is highly sensitive to the details of the hydrodynamic model.

Plasmas dominated by collisions are described by a pressure tensor that does not deviate far from isotropy, whereas an intense and ultrashort laser pulse propagating through an underdense plasma will drive the plasma anisotropically over typical acceleration timescales. Thus, it is important to accommodate 3-dimensionality and allow for anisotropy when investigating wave-breaking limits. The sensitivity of the wave-breaking limit to the details of the plasma model suggests that it could depend on the anisotropy of the pressure tensor.

One method for investigating the wave-breaking limit of a collisionless anisotropic plasma is to employ the warm plasma closure of velocity moments of the 1-particle distribution f satisfying the Vlasov-Maxwell equations [7]. Successive order moments of the Vlasov equation induce an infinite hierarchy of field equations for the velocity moments of f and at each finite order the number of unknowns is greater than the number of field equations. The warm plasma closure scheme sets the number of unknowns equal to the number of field equations by assuming that the terms containing the third order centred moment are negligible relative to those including second, first and zeroth order moments.

Our aim is to uncover the relationship between wavebreaking and the shape of f. In general, the detailed structure of f cannot be reconstructed from a few low-order moments so we adopt a different approach based on a particular class of piecewise constant 1-particle distributions. Our choice of distribution, although somewhat artificial, reduces the Vlasov equation to that of a boundary and combined with the Maxwell equations yields an integral for the wave-breaking limit in terms of the shape of the boundary.

Our approach may be considered as a multi-dimensional generalization of the 1-dimensional relativistic 'water-bag' model employed in [5] and details of the derivation of the formulae given here may be found in [9].

Units are used in which the speed of light $c = 1$ and the permittivity of the vacuum $\varepsilon_0 = 1$.

Electrostatic oscillations

We are interested in the evolution of a thermal plasma over timescales during which the motion of the ions is negligible in comparison with the motion of the electrons. We assume that the ions are at rest and distributed homogeneously with constant number density nion in the laboratory frame. Rather than attempting to solve the Vlasov-Maxwell system in its full generality we adopt a simple class of distributions that captures some of the properties of plasmas near wave-breaking.

In what follows, a point in Minkowski spacetime is denoted $x = (x^0, x^1, x^2, x^3)$ where x^0 is inertial time in the laboratory frame and (x^1, x^2, x^3) are Cartesian coordinates. An electron at the event x with relativistic 3-momentum $p = (p^1, p^2, p^3)$ is represented by the point $(x, \dot{x} = p/m)$ in 'relativistic phase space' where $\dot{x} = (\dot{x}^1, \dot{x}^2, \dot{x}^3)$. We consider distributions for which $f = \alpha$ is a positive constant inside some region of 'relativistic phase space' and $f = 0$ outside. In particular, the boundary of the support of the map $\dot{x}$ to $f(x, \dot{x})$ (the "water-bag") is topologically equivalent to the 2-sphere. We are interested in the behaviour of non-linear electrostatic oscillations along x^3, and to proceed further we employ distributions axisymmetric about $\dot{x}^3$ whose pointwise dependence in spacetime is on the wave's phase $\zeta = x^3 - vx^0$ only, where $0 < v < 1$ (this is sometimes called the 'quasi-static assumption').

Points on the boundary of the 'water-bag' are labelled locally by $\xi = (\xi^1, \xi^2)$, where $0 < \xi^1 < \pi$, $0 \le \xi^2 < 2\pi$. The shape of this boundary is specified by a single function $A(\xi^1)$, while its transverse extent is governed by a constant $R > 0$. The evolution of the boundary is described by a function $\mu(\zeta)$ over spacetime, and describes electrons with energy in the wave frame $m(\mu+A)/\gamma$, where m is the electron's rest mass and $\gamma = 1/\sqrt{1 - v^2}$ is the Lorentz factor of the wave's phase speed v. For the model discussed here, no electron described by f is moving faster along x^3 than the wave and wave-breaking occurs when any electron 'catches up' with the wave.

It may be shown [9] that, in the absence of external electromagnetic fields and without further approximation, the Vlasov-Maxwell system leads to

$$\frac{1}{\gamma^2}\frac{d^2\mu}{d\zeta^2} = -\frac{q^2}{m}n_{\rm ion}\gamma^2 - \frac{q^2}{m}2\pi R^2\alpha\int_0^\pi\Big([\mu + A(\xi^1)]^2 - \gamma^2[1+R^2\sin^2(\xi^1)]\Big)^{1/2}\sin(\xi^1)\,\cos(\xi^1)\,d\xi^1 \quad (1)$$

and

$$R^2\alpha\int_0^\pi A(\xi^1)\,\sin(\xi^1)\,\cos(\xi^1)\,d\xi^1 = -n_{\rm ion}\gamma^2 v. \quad (2)$$

where $q < 0$ is the charge on the electron.

The form of the 2nd order autonomous non-linear ordinary differential equation (1) for μ is fixed by specifying the generator $A(\xi^1)$ of the 'water-bag' subject to the normalization condition (2).

Electrostatic wave-breaking

The form of the integrand in (1) ensures that the magnitude of oscillatory solutions to (1) cannot be arbitrarily large. For our model, the wave-breaking value μ_{wb} is the largest μ for which the argument of the square root in (1) vanishes,

$$\mu_{\rm wb} = \max\Big\{-A(\xi^1) + \gamma\sqrt{1+R^2\sin^2(\xi^1)}\ \Big|\ 0\le\xi^1\le\pi\Big\}, \quad (3)$$

because $\mu < \mu_{wb}$ yields an imaginary integrand in (1) for some ξ^1. The positive square root in (3) is chosen because $m(\mu+A)$ is the relativistic energy of electrons described by the 'water-bag' and therefore positive, and it follows $\mu_{wb} + A(\xi^1) > 0$.

The electric field has only one non-zero component E (in the x^3 direction) where

$$E = \frac{m}{q}\frac{1}{\gamma^2}\frac{d\mu}{d\zeta} \quad (4)$$

and the wave-breaking limit E_{max} is obtained by evaluating the first integral of (1) between μ_{wb} where E vanishes and the equilibrium[1] value μ_{eq} of μ where E is at a maximum. Using (2) to eliminate α it follows μ_{eq} satisfies

$$\frac{1}{v}\int_0^\pi A(\xi^1)\sin(\xi^1)\cos(\xi^1)\,d\xi^1 = \int_0^\pi\Big([\mu_{\rm eq} + A(\xi^1)]^2 - \gamma^2[1+R^2\sin^2(\xi^1)]\Big)^{1/2}\sin(\xi^1)\cos(\xi^1)d\xi^1 \quad (5)$$

with

$$\int_0^\pi A(\xi^1)\sin(\xi^1)\cos(\xi^1)\,d\xi^1 < 0 \quad (6)$$

since α, $v > 0$. Equation (1) yields the maximum value E_{max} of E,

$$E^2_{\rm max} = 2mn_{\rm ion}\Bigg[-\mu_{\rm eq} + \mu_{\rm wb} + \frac{v}{\int_0^\pi A(\xi^1)\sin(\xi^1)\cos(\xi^1)d\xi^1}\int_{\mu_{\rm wb}}^{\mu_{\rm eq}}\int_0^\pi\Big([\mu + A(\xi^1)]^2 - \gamma^2[1+R^2\sin^2(\xi^1)]\Big)^{1/2}\sin(\xi^1)\cos(\xi^1)d\xi^1\,d\mu\Bigg]. \quad (7)$$

Example

To illustrate use of the above, we determine a wave-breaking limit for a nearly cold plasma whose distribution's transverse extent is much larger than its longitudinal extent.

Let $A(\xi^1) = -a\cos(\xi^1)$ where a is a positive constant characterising the longitudinal extent of the distribution. Using (7) it follows

$$E^2_{\rm max} = 2mn_{\rm ion}\Bigg[-\mu_{\rm eq} + \mu_{\rm wb} + \frac{3}{2}\frac{v}{a}\int_{\mu_{\rm wb}}^{\mu_{\rm eq}}\int_{-1}^{1}\Big([\mu + a\chi]^2 - \gamma^2[1+R^2(1-\chi^2)]\Big)^{1/2}\chi\,d\chi\,d\mu\Bigg] \quad (8)$$

where $\chi = -\cos(\xi^1)$ and equation (5) yields

$$\frac{3}{2}\frac{v}{a}\int_{-1}^{1}\Big([\mu_{\rm eq} + a\chi]^2 - \gamma^2[1+R^2(1-\chi^2)]\Big)^{1/2}\chi\,d\chi = 1. \quad (9)$$

Equation (3) may be written

$$\mu_{\rm wb} = \max\Big\{-a\chi + \gamma\sqrt{1+R^2(1-\chi^2)}\ \Big|\ -1\le\chi\le 1\Big\}, \quad (10)$$

and for a,R,γ satisfying

$$\frac{a}{R}\sqrt{\frac{1+R^2}{a^2+\gamma^2R^2}} < 1, \qquad R > 0 \quad (11)$$

[1] Note that the equilibrium of μ need not coincide with the plasma's thermodynamic equilibrium.

the maximum of $h(\chi) = -a\chi + \gamma = 1/\sqrt{1 + R^2 (1 - \chi^2)}$ over $-1 \le \chi \le 1$ coincides with a turning point of h. It follows

$$\mu_{\rm wb} = \frac{1}{R}\sqrt{(1+R^2)(a^2+\gamma^2R^2)}. \qquad (12)$$

For a ≪ R ≪ 1 equations (8, 9, 12) yield

$$E^2_{\rm max} \approx \frac{2m^2c^2\omega_p^2}{q^2}\left(\gamma - 1 - \frac{3}{4}\frac{v}{c}\gamma R\right) \qquad (13)$$

where $mcw_p\sqrt{2(\gamma - 1)}/|q|$ is the usual relativistic cold plasma wave-breaking limit of E (see, for example,[10]) and $w_p = \sqrt{n_{\rm ion}q^2/(m\varepsilon_0)}$ is the plasma angular frequency. Note that the speed of light c and the permittivity ε_0 of the vacuum have been restored. The parameter R may be eliminated in favour of an effective transverse 'temperature' $T_{\perp\rm eq}$ defined as

$$T_{\perp\rm eq} = \frac{1}{2k_B\, n_{\rm ion}}(P^{11}_{\rm eq} + P^{22}_{\rm eq}), \qquad (14)$$

where $P^{11}_{\rm eq}$, $P^{22}_{\rm eq}$ are components of the relativistic pressure tensor in the laboratory frame derived from the 'equilibrium' water-bag given by $\mu = \mu_{\rm eq}$ (see footnote 1 on p. 111) and k_B is Boltzmann's constant. It follows

$$R \approx \sqrt{\frac{5k_BT_{\perp\rm eq}}{mc^2}} \qquad (15)$$

where the speed of light c has been restored.

Conclusion

We have developed a method for investigating the relationship between the shape of a 1-particle distribution and electrostatic non-linear thermal plasma waves near breaking. An approximation to the wave-breaking limit of the electric field was obtained for a particular axisymmetric distribution.

Further analysis of (7, 5, 3) will be presented elsewhere.

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