Wave-breaking limits for non-quasi-static oscillations in a warm 1-D electron plasma

Contact **raoul.trines@stfc.ac.uk**

R. M. G. M. Trines

Central Laser Facility, STFC, Rutherford Appleton Laboratory, HSIC, Didcot, Oxon OX11 0QX, UK

In this work, wave breaking for general, non-quasi-static oscillations in warm plasma is investigated using Lagrangian methods. In particular, the effects of secular behaviour on wave breaking limits are explored, and it is shown that thermal effects can sometimes prevent wave breaking by curbing secular behaviour. The oscillation equations for fully relativistic warm plasma are cast into Lagrangian form, and wave breaking limits are derived for waves in warm plasma having non-constant density. These results have important applications in electron acceleration schemes that employ a wakefield or a slow beat wave propagating down a density gradient.

The problem of wave breaking of one-dimensional quasi-static waves, i.e. waves that are nonevolving with respect to a comoving coordinate $\xi = x - v_\varphi t$ for some fixed speed v_φ, has been solved completely [1,2,3,4,5,6]. However, there are many configurations of non-quasi-static waves for which it is also important to know the wave breaking limits. Wave breaking in plasmas having non-constant background densities, for example, is of much importance to laser resonance absorption [7,8], electron acceleration schemes using colliding laser pulses on downward density slopes [9,10], and two-stage injection-acceleration schemes where electrons are injected into a laser-driven wakefield at high plasma density and subsequently accelerated at a lower density [11]. In a recent experiment [12], electron bunches having very low absolute longitudinal momentum spread (0.17 MeV/c) have been produced by wave breaking of a plasma wave propagating down a density gradient, as predicted by Bulanov *et al.* [13]. Furthermore, a plasma oscillation may have a spatially varying amplitude for any number of reasons, and this has its own peculiar effects on wave breaking if the oscillation is relativistic [14]. While various cases of breaking of non-quasi-static waves have been investigated for cold plasma [1,13,14,15,16], there are hardly any results for such waves in the presence of thermal effects. Since a realistic plasma needs to have a finite temperature to prevent recombination, it is important that the theory of non-quasi-static plasma oscillations is expanded to include thermal effects also.

The wave breaking limits for quasi-static waves are well-established. For a quasi-static wave with amplitude A, frequency ω and wave number k, cold-plasma wave breaking sets in at $kA = 1$ or $v_\varphi \equiv \omega/k$ [1,2]. Warm-plasma wave breaking occurs at $v = v_\varphi[1-\alpha k_B T/(m v_\varphi^2)^{1/(\alpha+1)}$ for non-relativistic plasma [3,4] [where $\alpha = 3$ ($\alpha = 1$) denotes adiabatic (isothermal) compression], or $v = v_\varphi[1-\sqrt{\beta/\gamma_\varphi^2+\sqrt{\beta/(\gamma_\varphi^3 v_\varphi)}}]/(1+v_\varphi\sqrt{\beta}/\gamma_\varphi)$ for relativistic plasma [5,6] (where $\beta = 3k_B T/(m_e c^2)$ and $\gamma_\varphi^2 = 1/(1 - v_\varphi^2)$). For non-quasi-static waves however there is some controversy between the models of Coffey [3] and Infeld and Rowlands [17], which must be resolved before one can proceed. Where Coffey's model predicts that wave breaking occurs when the mean plasma velocity v satisfies

$$v = v_\varphi[1-(\alpha v_\varphi^2/(v_\varphi^2)^{1/(\alpha+1)}] \qquad (1)$$

the paper by Infeld and Rowlands claims that wave breaking occurs simply when $v = v_\varphi$, i.e. the same condition as found for a cold plasma [1], independent of the plasma temperature. However, a close scrutiny of Ref. [17] reveals an algebraic error and an error in their reasoning; correction of these errors leads to a recovery of Coffey's limit (1). A detailed discussion can be found in Ref. [18].

Now that the proper wave breaking limit for non-quasi-static plasma oscillations has been determined, one can proceed to the study of secular behaviour. Secular behaviour is the phenomenon that the phase difference between neighbouring fluid elements in a plasma oscillation is not constant in time. It occurs when the oscillation frequency of the plasma electrons depends on $\bar{x}$ [1,14,15,16,13]. This happens, for example, in an inhomogeneous plasma [1,12,13] or for a relativistic plasma oscillation in homogeneous plasma where the oscillation frequency can still depend on $\bar{x}$ through the oscillation amplitude [14,16]. In a cold plasma, secular behaviour will cause a plasma oscillation to eventually break at any amplitude. This works as follows. A plasma oscillation having a position-dependent frequency is given by e.g. $x(\bar{x},\tau) = A\cos[k_0\bar{x} - \omega(\bar{x})\tau]$. The effective wave number k_{eff} is then derived from $\partial x/\partial\bar{x}$: $k_{\mathrm{eff}} = k_0 - (\partial\omega/\partial\bar{x})\tau$, i.e. k_{eff} grows linearly in time. Wave breaking will occur when $|k_{\mathrm{eff}}A| = 1$, so even for $k_0 A = 1$ secular behaviour will cause the wave to break after a time of at most $\tau_{\mathrm{WB}} = 1/(A|\partial\omega/\partial\bar{x}|)$. Also, by defining the effective phase speed as $v_{\varphi,\mathrm{eff}} = \omega k_{\mathrm{eff}}$ [13], it follows that wave breaking occurs if the peak forward fluid speed $v_{\max}$ satisfies $v_{\max} = \omega A = v_{\varphi,\mathrm{eff}} k_{\mathrm{eff}} A = v_{\varphi,\mathrm{eff}}$. As shown in Ref. [13], secular behaviour will cause the wave's phase speed to decrease until it equals the peak forward fluid speed, at which point the wave breaks. Electron trapping by a wakefield on a downward density ramp, as demonstrated by Geddes *et al.* [12], is based on this principle.

While the role of secular behaviour in cold plasma wave breaking is well studied [1,13,15,14,16], secular behaviour in warm plasma wave breaking is only touched upon by Infeld and Rowlands [19]. Even so, a linearised version of the warm-plasma wave equation is used in Ref. [19], which leads to an incorrect wave breaking limit because it underestimates the plasma pressure near breaking. Because of this, and because the combination of secular and thermal effects yields some surprising results, this subject will be studied here.

Although thermal effects will normally reduce the wave breaking amplitude [5,4,3,6], they may surprisingly also delay or prevent the onset of wave breaking in the case of secular behaviour. This is because secular behaviour will make k grow, while thermal effects will make k advect, so the regions of largest k and largest $\partial k/\partial t$ will no longer coincide. The secular growth of k will then saturate eventually, preventing the onset of wave breaking in certain circumstances. As an example, the evolution of k will be investigated in a thermal plasma, on a finite slope where the background density n_0 falls an amount $\Delta n>0$ over a length L, as used e.g. in electron trapping experiments by Geddes *et al.* [12]. From $\partial k/\partial\tau + \partial\omega/\partial\bar{x} = 0$ [15] and the Bohm-Gross dispersion relation $\omega^2 = \omega_p^2(\bar{x}) + \alpha v_T^2 k^2$, it is found that (assuming that $k\lambda_D = 1$ everywhere):

$$\frac{\partial k}{\partial\tau} + \frac{\alpha v_T^2}{\omega_p} k \frac{\partial k}{\partial\bar{x}} = -\frac{\omega_p}{2n_0}\frac{\partial n}{\partial\bar{x}}. \qquad (2)$$

A second "Lagrangianisation" $\tau' = \tau$, $\bar{x}' = \bar{x} - \alpha\lambda_D^2\omega_p\int k\,d\tau'$ [19] yields that $\partial k_{\text{eff}}/\partial\tau' = -(\omega_p/2n_0)\partial n/\partial\bar{x} \approx (\omega_p/2n_0)(\Delta n/L)$. Using $\partial\bar{x}/\partial\tau' = \alpha\lambda_D^2\omega_p k$, this expression integrates to $\Delta(k_{\text{eff}}\lambda_D^2) \leq \Delta n/(\alpha n_0)$ over the entire length of the slope, or $k_{\text{eff}}\lambda_D \leq k_{\max}\lambda_D \equiv \sqrt{(k_0\lambda_D)2 + \Delta n/(\alpha n_0)}$, where $k = k_0$ at $\tau = 0$. As in (1), wave breaking then occurs if $k_{\max}A;1 - (\alpha k_{\max}^2\lambda_D^2)^{1/(1+\alpha)}$ (increasing the wave number lowers the wave breaking limit in two ways). The wave-breaking amplitude A_{WB} and corresponding electric field E_{WB} are then given by:

$$A_{\text{WB}} = \frac{\lambda_D\{1-[\alpha(k_0\lambda_D)^2 + \Delta n/n_0]^{1/(1+\alpha)}\}}{\sqrt{(k_0\lambda_D)^2 + \Delta n/(\alpha n_0)}}, \qquad (3)$$

$$E_{\text{WB}} = v_\varphi\{1 - C_\alpha[\alpha(k_0\lambda_D)^2 + \Delta n/n_0]^{1/(1+\alpha)}\}, \qquad (4)$$

where $v_\varphi = \omega(k_{\max})/k_{\max}$ is the local phase speed of the wave (using Bohm-Gross for $\omega(k)$), $C_1 = 1$ [4] and $C_3 = 4/3$ [3]. For a homogeneous plasma, $\Delta n = 0$ and $A_{\text{WB}} = [1 - (\alpha k_0^2\lambda_D^2)^{1/(1+\alpha)}]/k_0$, a result already known from the work of Dawson [1] and Coffey [3]. For an inhomogeneous cold plasma, $\Delta n = 0$ and $\lambda_D = 0$, leading to $A_{\text{WB}} = 0$. This is a consequence of the fact that the electron oscillations in an inhomogeneous cold plasma exhibit secular behaviour [13]: k grows linearly in time, and no matter how small the amplitude, $kA\uparrow 1$ and the wave will eventually break. For both $\Delta n = 0$ and $\lambda_D = 0$ however, $A_{\text{WB}} > 0$ again, provided that $\alpha(k_0\lambda_D)^2 + \Delta n/n_0 < 1$. It follows that the secular behaviour that is caused by the plasma inhomogeneity is curbed by thermal effects: k will only grow a finite instead of an unlimited amount, and for sufficiently small A and Δn, wave breaking will not happen in spite of the density ramp.

The above results can be extended to cover plasma oscillations having relativistic amplitudes and/or phase speeds, as encountered in e.g. laser-wakefield acceleration [12]. A full analysis is beyond the scope of this manuscript; the reader is referred to Ref. [18].

In summary, wave breaking has been studied for general, non-quasi-static plasma waves. It has been found that breaking of such waves has many traits in common with the breaking of quasi-static waves, confirming the notion that quasi-static wave breaking is a proper special case of general wave breaking. A proper investigation of secular behaviour in thermal plasma has been carried out here for the first time. It has been found that thermal effects can curb secular behaviour and prevent wave breaking in certain specific circumstances, even though thermal effects normally facilitate wave breaking [3,4,5,6]. The wave breaking limits for plasma waves in inhomogeneous thermal plasma have been derived for the first time. As such, this work is a generalisation of earlier work on breaking of quasi-static waves in thermal plasma [6], with important consequences for the study of inhomogeneous plasma oscillations or plasma waves in inhomogeneous plasma. The author would like to thank P. Norreys for helpful comments. This work was supported by the STFC Accelerator Science and Technology Centre (ASTEC).

References

1. J. M. Dawson, *Phys. Rev*. **113**, 383 (1959).
2. A. I. Akhiezer and R. V. Polovin, *Zh. Exp. Teor. Fiz*. **30**, 915 (1956) [*Sov. Phys. JETP* **3**, 696 (1956)].
3. T. P. Coffey, *Phys. Fluids* **14**, 1402 (1971).
4. E. Infeld and G. Rowlands, *J. Phys. A* **12**, 2255 (1979).
5. T. Katsouleas and W. B. Mori, *Phys. Rev. Lett*. **61**, 90 (1988).
6. R. M. G. M. Trines and P. A. Norreys, *Phys. Plasmas* **13**, 123102 (2006).
7. B. Bezzerides and S. J. Gitomer, *Phys. Fluids* **26**, 1359 (1983).
8. A. Bergmann and P. Mulser, *Phys. Rev. E* **47**, 3585 (1993).
9. J. Faure *et al.*, *Nature* **444**, 737 (2006).
10. G. Fubiani, E. Esarey, C. B. Schroeder, and W. P. Leemans, *Phys. Rev. E* **73**, 026402 (2006).
11. P. Sprangle *et al.*, *Phys. Rev. E* **63**, 056405 (2001).
12. C. G. R. Geddes *et al.*, *Phys. Rev. Lett*. **100**, 215004 (2008).
13. S. Bulanov, N. Naumova, F. Pegoraro and J. Sakai, *Phys. Rev. E* **58**, R5257 (1998).
14. J. F. Drake, Y. C. Lee, K. Nishikawa and N.L . Tsintsadze, *Phys. Rev. Lett*. **36**, 196 (1976).
15. G. B. Whitham, *Linear and Nonlinear Waves*, (Wiley, New York, 1974).
16. S. V. Bulanov *et al.*, *Phys. Fluids B* **4**, 1935 (1992).
17. E. Infeld and G. Rowlands, *Phys. Rev. Lett*. **58**, 2063 (1987).
18. R. Trines, *Phys. Rev. E* 79, 056406 (2009).
19. E. Infeld and G. Rowlands, *Phys. Rev. A* **42**, 838 (1990).